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INTERFACIAL ELASTIC PARAMETERS IN TORSIONAL VIBRATIONS OF A PERIODIC STRUCTURED CYLINDRICAL ROD

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Abstract—A method is developed for qualitatively sketching the frequency spectrum of torsional waves propagating in a solid cylinder with periodic structure. This method introduces an interfacial elastic parameter, analogous to an impedance, which simplifies the problem considerably.

INTRODUCTION

The analysis of wave propagation in periodic elastic structures, i.e. structures in which the elastic constants of the material vary in a periodic manner, is of great interest due to the presence of passing and stopping frequency bands in such structures. While such an analysis is in general quite complicated, closed form solutions have recently been obtained for a number of simpler problems. It is the purpose of this study to discuss an alternative approach to such problems which may be of some value in gaining physical insight into the properties of the solution and in qualitatively "sketching" the frequency spectrum with reduced effort.

FORMULATION OF BASIC PROBLEM

One common physical case studied so far is that of torsional elastic wave propagation in a bi-element composite circular cylinder, either solid, Ref. [1], or hollow, Ref. [2]. The outer (and inner, if present) boundaries are taken as stress free. The prismatic cylinder is assumed to be infinite in length, $|z| < \infty$, and consists of two homogeneous elements, comprising region I of length l, density ρ and shear modulus μ and region II of length l', density ρ' and shear modulus μ', repeated periodically in the axial (z) direction with a unit cell size $d \equiv (l + l')$. Such a problem has only a single displacement component, U_θ, independent of the θ coordinate in a cylindrical coordinate system (r, θ, z). This displacement component satisfies a wave equation with phase speed $c = (\mu/\rho)^{1/2}$ and $c' = (\mu'/\rho')^{1/2}$ in regions I and II respectively. We choose the z coordinate positive to the right and assume a time harmonic dependence proportional to exp $(-i\omega t)$, where ω is the angular frequency in radians per unit of time.

In this simple example, the governing field equation readily separates in (r, z) to give tangential component of displacement in the form:

(a) Region I, $0 < z < l$:

$$U_\theta^I = [A \sin(\lambda z/l) + B \cos(\lambda z/l)] Z^I(\kappa r/a) \exp(-i\omega t), \tag{1}$$

(b) Region II, $-l' < z < 0$;

$$U_\theta^{II} = [A' \sin(\lambda' z/l') + B' \cos(\lambda' z/l')] Z^{II}(\kappa' r/a) \exp(-i\omega t).$$

We take the outer radius of the cylinder to be a, the inner radius (if one exists) b, the half thickness $h = (a - b)/2$ and $\Omega = \omega/\omega_s$ where $\omega_s = \pi c/2h$ is the lowest thickness shear frequency of an infinite, isotropic, homogeneous plate of half thickness, h, mass density, ρ, and shear modulus, μ. We have condensed the notation (which corresponds to [1] and [2]) by introducing $\kappa \equiv \pi ka/2h$, $\kappa' \equiv \pi k'a/2h$, $\lambda \equiv \pi\xi l/2h$, $\lambda' \equiv \pi\xi' l'/2h$. The radial solutions, Z, are given by

$$\begin{aligned} Z^I &= (2/\kappa) J_1(\kappa r/a) \qquad && \kappa^2 \geq 0 \\ &= (2/\kappa) I_1(\kappa r/a) && \kappa^2 \leq 0 \\ &= r/a && \kappa^2 = 0 \end{aligned} \tag{2}$$

for the solid cylinder and

$$
\begin{aligned}
Z^I &= (2/\kappa)J_1(\kappa r/a) - (\pi\kappa/2)DY_1(\kappa r/a) && \kappa^2 \geq 0 \\
&= (2/\kappa)I_1(\kappa r/a) + \kappa DK_1(\kappa r/a) && \kappa^2 \leq 0 \\
&= r/a + Da/r && \kappa^2 = 0
\end{aligned} \tag{3}
$$

for the hollow cylinder with similar equations for Z^{II} in terms of κ' and D' and where

$$
\begin{aligned}
(l\kappa/a)^2 + \lambda^2 &= (l\omega/c)^2 \quad \text{or} \quad k^2 + \xi^2 = \Omega^2, \\
(l'\kappa'/a)^2 + \lambda'^2 &= (l'\omega/c')^2 \quad \text{or} \quad k'^2 + \xi'^2 = (c/c')^2\Omega^2.
\end{aligned} \tag{4}
$$

The radial boundary conditions require zero shear stress, $\tau_{r\theta} = \mu r\partial[U_\theta/r]/\partial r$, on the outer surface $r = a$ and either zero shear stress on the inner surface $r = b$ for the hollow cylinder or a bounded displacement at the origin 0^+ for the solid cylinder (this was already implied in the form of Z used above for the solid cylinder).

For the solid cylinder, the radial boundary condition requires

$$J_2(\kappa) = 0, \tag{5}$$

for $\kappa^2 \geq 0$ and has no roots for $\kappa^2 < 0$. Roots of this transcendental equation as given in [3] are $\kappa_0 = 0$, $\kappa_1 = 5.1356$, $\kappa_2 = 8.4172$, etc.

For the hollow cylinder, the boundary conditions require

$$
\begin{aligned}
J_2(\kappa)Y_2(\kappa t) - J_2(\kappa t)Y_2(\kappa) = 0, \qquad t \equiv b/a \\
D \equiv (\pi\kappa^2/4)Y_2(\kappa)/J_2(\kappa),
\end{aligned} \tag{6}
$$

for $\kappa^2 > 0$ and no solution for $\kappa^2 < 0$. Roots of this transcendental equation must be calculated as a function of t. For $t = 1/3$, $\kappa_1 = 1.1892$, $\kappa_2 = 2.1160$, $\kappa_3 = 3.0811$, etc. and for $t = 1/2$, $\kappa_1 = 1.08446$, $\kappa_2 = 2.04602$, $\kappa_3 3.03122$, etc. Similar results hold for κ'.

The remaining boundary conditions apply to surfaces of constant z. The origin and direction of the z axis are arbitrary; we choose $z = 0$ at the interface between regions II and I and first require continuity of U_θ and $\tau_{z\theta} = \mu\partial U_\theta/\partial z$. This leads to the conditions

$$\nu A = \nu' A', \qquad B = B', \tag{7}$$

where

$$\nu \equiv \mu\lambda/l \quad \text{and} \quad \nu' \equiv \mu'\lambda'/l.$$

The remaining interfacial condition is a quasi-periodicity condition which, by Floquet's theorem, requires the wave amplitude to have the same periodic structure as that of the elastic medium, in order to have a solution

$$U_\theta(r, z, t) = w(r, z)\exp i(\gamma z - \omega t),$$

where w is periodic in z with periodicity d, i.e., $w(r, z + d) = w(r, z)$ and γ is the Floquet's wave number corresponding to the phase shift, which has to be determined from the solution of the problem.

This leads to the quasi-periodic conditions

$$
\begin{aligned}
&\text{(a)} \qquad U_\theta^I(r, l, t) = U_\theta^{II}(r, -l', t)\exp i\gamma d, \\
&\text{(b)} \qquad \tau_{z\theta}^I(r, l, t) = \tau_{z\theta}^{II}(r, -l', t)\exp i\gamma d,
\end{aligned} \tag{8}
$$

where for convenience we have dropped explicit time harmonic dependence $\exp(-i\omega t)$. Equation (8) implies $Z^{I}(\kappa r/a) \equiv Z^{II}(\kappa' r/a)$, i.e. $\kappa \equiv \kappa'$; and

$$\text{(a)} \qquad A \sin \lambda + B \cos \lambda = [-A' \sin \lambda' + B' \cos \lambda'] \exp i\gamma d,$$

$$\text{(b)} \qquad \nu[A \cos \lambda - B \sin \lambda] = \nu'[A' \cos \lambda' + B' \sin \lambda'] \exp i\gamma d. \tag{9}$$

INTERFACIAL PARAMETER

While the problem is completely formulated at this point and has indeed been solved in this form in Refs. [1,2], we introduce a slight modification in the variables of the problem

We introduce the "interfacial elastic parameter", Q, as the ratio of stress to displacement at an interface. Clearly Q is continuous throughout the rod. Consider first $Q|_{z=0} \equiv Q_0$ defined as

$$Q_0 = \tau_{z\theta}(0)/U_\theta(0) = \nu A/B = \nu' A'/B'. \tag{10}$$

Equations (9a) and (9b) may then be rewritten as

$$\text{(a)} \qquad B(Q_0 \sin \lambda + \nu \cos \lambda) = B'(\nu/\nu')[-Q_0 \sin \lambda' + \nu' \cos \lambda'] \exp(i\gamma d),$$

$$\text{(b)} \qquad B(Q_0 \cos \lambda - \nu \sin \gamma) = B'[Q_0 \cos \lambda' + \nu' \sin \lambda'] \exp(i\gamma d). \tag{11}$$

Consistency of these two homogeneous equations requires that the determinant of the coefficients be zero. This leads us to the frequency equation

$$Q_0^2[\nu' \sin \lambda \cos \lambda' + \nu \sin \lambda' \cos \lambda] + Q_0(\nu'^2 - \nu^2) \sin \lambda \sin \lambda' + \nu\nu'(\nu' \sin \lambda' \cos \lambda + \nu \sin \lambda \cos \lambda') = 0, 0 \leq Q_0 < \infty. \tag{12}$$

On the other hand, we may eliminate Q_0 from eqn (11) by using eqn (10) and then obtain the usual dispersion equation.

$$[\exp(i\gamma d)]^2 + [(\nu/\nu' + \nu'/\nu) \sin \lambda \sin \lambda' - 2 \cos \lambda \cos \lambda'] \exp(i\gamma d) + 1 = 0, \tag{13}$$

which relates Floquet wave number γ to the axial wave numbers λ and λ'. Clearly, the second order equation has two solutions whose product is unity and whose sum is

$$\cos \gamma d = \cos \lambda \cos \lambda' - \frac{1}{2}\left(\frac{\nu}{\nu'} + \frac{\nu'}{\nu}\right) \sin \lambda \sin \lambda'. \tag{14}$$

If we examine the completely homogeneous case, $l = l'$, $\mu = \mu'$, $\rho = \rho'$, $\xi = \xi'$ and $\nu = \nu'$, we get the solutions $\gamma = \pm(\pi/d)(\xi l/h \pm 2n)$, $n = 0, 1, 2, \ldots$. On the extended zone scheme we select $n = 0$, and therefore the two values for $\gamma = \pm \pi\xi l/2h = \pm\lambda/l$ are real and simply represent the two directions in which torsional waves can propagate with no change in amplitude. This is best seen by examining the form of the z dependence in the tangential component of displacement

$$U_\theta = B[(A/B) \sin(\lambda z/l) + \cos(\lambda z/l)]Z(\kappa r/a),$$

where, from eqn (10) it can easily be shown that

$$A/B = Q_0/\nu = \pm i.$$

Hence, as expected

$$U_\theta = B[\cos(\lambda z/l) \pm i \sin(\lambda z/l)]Z(\kappa r/a),$$

$$= B \exp(\pm i\lambda z/l)Z(\kappa r/a),$$

and represents the two directions in which the torsional wave can propagate without change of form.

SPECIAL VALUES FOR INTERFACIAL PARAMETER

While there may be no general advantage in calculating Q_0 for a given Ω and then calculating γ etc. there are some special circumstances where the introduction of Q_0 is useful. Consider first the limiting cases when Q_0 tends to 0 or ∞, corresponding to zero stress or zero displacement, respectively, at $z = 0$. The former, $Q_0 = 0$ leads to (if $\cos\lambda$ and $\cos\lambda'$ are non-zero)

$$\begin{aligned}
&\text{(a)} && \nu\tan\lambda + \nu'\tan\lambda' = 0, \qquad \nu,\nu' \neq 0\\
&\text{(b)} && \cos\gamma d = \frac{1}{2}(\cos\lambda'/\cos\lambda + \cos\lambda/\cos\lambda'),\\
&\text{(c)} && U_\theta{}^I = B\cos(\lambda z/l)Z(\kappa r/a),\\
&\text{(d)} && U_\theta{}^{II} = B\cos(\lambda' z/l')Z(\kappa r/a).
\end{aligned} \tag{15}$$

Similarly, for $Q_0 = \infty$, we have (if $\sin\lambda$ and $\sin\lambda'$ are non-zero)

$$\begin{aligned}
&\text{(a)} && \nu\cot\lambda + \nu'\cot\lambda' = 0, \qquad \nu,\nu' \neq 0\\
&\text{(b)} && \cos\gamma d = \frac{1}{2}(\cos\lambda'/\cos\lambda + \cos\lambda/\cos\lambda'),\\
&\text{(c)} && U_\theta{}^I = A\sin(\lambda z/l)Z(\kappa r/a),\\
&\text{(d)} && U_\theta{}^{II} = A(\nu/\nu')\sin(\lambda' z/l')Z(\kappa r/a).
\end{aligned} \tag{16}$$

These frequency equations for $Q_0 = 0$ and ∞, are not only simpler to solve than the general set, they are formally of the same structure as the frequency equations for the endpoints of the Brillouin zones which define the passing and stopping bands. Equations at the zone ends are obtained by setting $\exp(i\gamma d) = \pm 1$ in eqn (13); these are the cut-off points where γ changes from real to imaginary values. For $\exp(i\gamma d) = +1$, this has solutions with a period d at frequencies defined by

$$\begin{aligned}
&\text{(a)} && \nu\tan(\lambda/2) + \nu'\tan(\lambda'/2) = 0,\\
&\text{(b)} && \nu\tan(\lambda'/2) + \nu'\tan(\lambda/2) = 0,
\end{aligned} \tag{17}$$

and for $\exp(i\gamma d) = -1$, it has solutions of period $2d$ at frequencies defined by

$$\begin{aligned}
&\text{(a)} && \nu\tan(\lambda/2) - \nu'\cot(\lambda'/2) = 0,\\
&\text{(b)} && \nu\cot(\lambda/2) - \nu'\tan(\lambda'/2) = 0.
\end{aligned} \tag{18}$$

We thus see that wave numbers satisfying eqn (15a) are one-half that satisfying eqn (17a); that for eqn (16a) are one-half the wave numbers satisfying eqn (17b).

It may be pointed out that Q_0 satisfies a quadratic equation, and for a fixed κ and given values of (Ω, γ), there are always two values of Q_0. Thus

$$Q_0{}^{(1)} = 0; \qquad Q_0{}^{(2)} = \nu\tan\lambda = -\nu'\tan\lambda', \tag{18a}$$

are the two roots of eqn (12) in the presence of the constraining eqn (15a). The displacement field in the second case is

$$\begin{aligned}
U_\theta{}^I &= A\csc\lambda\cos\lambda(z/l - 1)Z(\kappa r/a),\\
U_\theta{}^{II} &= -A'\csc\lambda'\cos\lambda'(z/l' + 1)Z(\kappa r/a).
\end{aligned} \tag{19a}$$

Similarly

$$Q_0{}^{(1)} = \infty; \qquad Q_0{}^{(2)} = -\nu\cot\lambda = \nu'\cot\lambda', \tag{18b}$$

are the two roots of the same equation in the presence of the constraining eqn (16a). The displacement field in the second case is

$$U_\theta^I = A \sec \lambda \sin \lambda(z/l - 1)Z(\kappa r/a),$$
$$U_\theta^{II} = A' \sec \lambda' \sin \lambda'(z/l' + 1)Z(\kappa r/a). \qquad (19b)$$

In each of these four cases the Floquet number γ is governed by the same eqn (15a).

These "second" solutions correspond to the same problem with the z origin shifted by one segment, e.g. by l. For example, the solution for $Q_0^{(1)} = 0$ corresponds to a zero shear stress at z equal to zero while the "second" solution, $Q_0^{(2)} = \nu \tan \lambda$, corresponds to a zero shear stress at $z = l$, etc.

The equation (15b = 16b) can be rewritten as

$$\sin(\gamma d/2) = \pm\frac{1}{2}((\cos\lambda/\cos\lambda')^{1/2} - (\cos\lambda'/\cos\lambda)^{1/2}),$$

$$\cos(\gamma d/2) = \pm\frac{1}{2}((\cos\lambda/\cos\lambda')^{1/2} + (\cos\lambda'/\cos\lambda)^{1/2}), \qquad (20)$$

and therefore we see that at the end points of the Brillouin zones

$$\cos\lambda' - \cos\lambda = 0 \qquad \text{when } \gamma d = 0, 2\pi, 4\pi, \ldots$$
$$\cos\lambda' + \cos\lambda = 0 \qquad \text{when } \gamma d = \pi, 3\pi, 5\pi, \ldots \qquad (21)$$

Thus on the left end of the zone

$$\lambda' = \lambda \pm \eta_e\pi_g, \quad n_e = 0, 2, 4, \ldots \qquad (22a)$$

and on the right end of the zone

$$\lambda' = \lambda \pm \eta_0\pi, \qquad \eta_0 = 1, 3, 5, \ldots \qquad (22b)$$

Consider now the case when $Q_0 = 0$. On the left end of the zone $\cos\lambda' = \cos\lambda$ implies $\sin\lambda' = \sin\lambda$ and therefore from eqn (15a)

$$(\nu + \nu')\tan\lambda = 0. \qquad (23)$$

Consequently, either $(\nu + \nu') = 0$, or $\sin\lambda = 0$. Hence, either

$$\lambda = \pm\pi\eta_e/[(\mu/\mu')(l'/l) + 1], \quad \text{or} \quad \lambda = m\pi, \quad m = 0, 1, 2, \ldots \qquad (24)$$

Knowing λ from this equation, we can now find λ' from eqn (22a). However, in the second case, η_e in eqn (22a) and m in eqn $(24)_2$, cannot both be arbitrary since from eqn (4)

$$\lambda'^2 = (l'/l)^2(c/c')^2\lambda^2 + (l'/a)^2\kappa^2((c/c')^2 - 1). \qquad (25)$$

When $Q_0 = \infty$, we find from eqns $(21)_1$ and (16a) that

$$(\nu + \nu')\cot\lambda = 0,$$

and therefore, either $(\nu + \nu') = 0$, or $\cos\lambda = 0$. Hence, in this case either

$$\lambda = \pm\pi\eta_e/[(\mu/\mu')(l'/l) + 1], \quad \text{or} \quad \lambda = m\pi/2, \quad m = 1, 3, 5, \ldots \qquad (26)$$

On the right end of the zone, we have simply to replace the integers η_e by η_0 to obtain corresponding formulas for λ and λ' when $Q_0 = 0$ or $Q_0 = \infty$.

It is easy to see that in the interior of the Brillouin zone $0 < \gamma d < \pi$, γd is always complex for real values of λ and λ', $\lambda \neq \lambda'$. This follows immediately from the fact that eqn (15b) can be rewritten as

$$\exp i\gamma d = \cos\lambda/\cos\lambda'. \tag{27}$$

We therefore conclude that when $Q_0 = 0$ (stress-free interface), or when $Q_0 = \infty$ (displacement-free interface) torsional waves cannot be propagated in a prismatic cylinder with periodic structure and are therefore damped out.

We now consider those values of Q_0 which lead to information concerning frequencies in the passing band. From eqn (12) we find that when $Q_0 = \pm i\nu'$

$$\nu'(\nu^2 - \nu'^2)\sin\lambda\exp(\pm i\lambda') = 0, \tag{28}$$

and when $Q_0 = \pm i\nu$

$$\nu(\nu^2 - \nu'^2)\sin\lambda'\exp(\pm i\lambda) = 0. \tag{29}$$

We assume that $\nu \neq \nu'$, $\nu \neq 0$ and $\nu' \neq 0$. Then for $Q_0 = \pm i\nu'$, $\sin\lambda = 0$ and in accordance with eqn (14) $\cos\gamma d = \pm\cos\lambda'$. Therefore

$$\lambda = \eta\pi, \qquad \eta = 0, 1, 2, \ldots \qquad \gamma d = \lambda' \pm m\pi, \qquad m = 0, 1, 2, \ldots \tag{30}$$

It therefore follows from eqn (4) that

$$\begin{aligned} \Omega^2 &= k^2 + (2\eta h/l)^2 \\ (c/c')^2\Omega^2 &= k^2 + (2h/\pi l')^2(\gamma d \pm m\pi)^2. \end{aligned} \tag{31}$$

For every value of radial wave number k, these two equations determine a unique value of Ω and γ. Plotting Ω vs γd, the first equation gives us lines of constant Ω for every given k and for different values of η. The second equation gives us a series of hyperbolas (or straight lines when $k_0 = 0$). Starting with the cut-off frequencies when $\gamma d = 0$, the real branches of the dispersion curves lie between the bounds defined by eqn (31). The dispersion curve will cross the bounds only at points of intersection, where $Q_0 = \pm\, i\nu'$.

Another independent set of bounds can similarly be found when $Q_0 = \pm i\nu$, leading to $\sin\lambda' = 0$. In this case the two equations of the bounds are

$$\begin{aligned} (c/c')^2\Omega^2 &= k^2 + (2\eta h/l')^2, \\ \Omega^2 &= k^2 + (2h/\pi l)^2(\gamma d \pm m\pi)^2. \end{aligned}$$

Since the cut-off frequencies at $\gamma = 0$ and $\gamma d = \pi$ are available from the roots of eqns (17) and (18), the knowledge of additional intersection points inside the Brillouin zone, provides us with sufficient information to sketch the dispersion curve qualitatively. The use of interfacial parameter Q_0 therefore provides us with a scheme for a qualitative solution, which may be valuable in more complex problems.

Physically, these two cases correspond to $A' = iB'$ and $A = iB$ respectively. These conditions in turn lead to the requirement that the ratio of shear stress, $\tau_{z\theta}(z)$, to displacement, $U_\theta(z)$, be uniform throughout region II for $Q_0 = i\nu$ and throughout region I for $Q_0 = i\nu'$, i.e. $\tau_{z\theta}{}^{II}(z)/U_\theta{}^{II}(z) = \tau_{z\theta}{}^{II}(0)/U_\theta{}^{I}(0)$ respectively.

The bounding curves for $Q_0 = i\nu$, and $i\nu'$, are shown in Fig. 1 along with the actual dispersion curve as calculated in Ref. [1] for the lowest branch, $\kappa_0 = 0$, for a solid cylinder. The parameters used are $\mu/\mu' = 1/40$, $l/a = 3/5$, $l'/a = 3$, $c/c' = 1/4$. For these parameters, the bounding curves for $Q_0 = i\nu'$ are $\Omega = 5\eta/3$; $(\gamma d/\pi) = (3/4)\Omega + \eta$ and for $Q_0 = i\nu$ are $\Omega = 4\eta/3$; $(\gamma d/\pi) = (3/5)\Omega + \eta$.

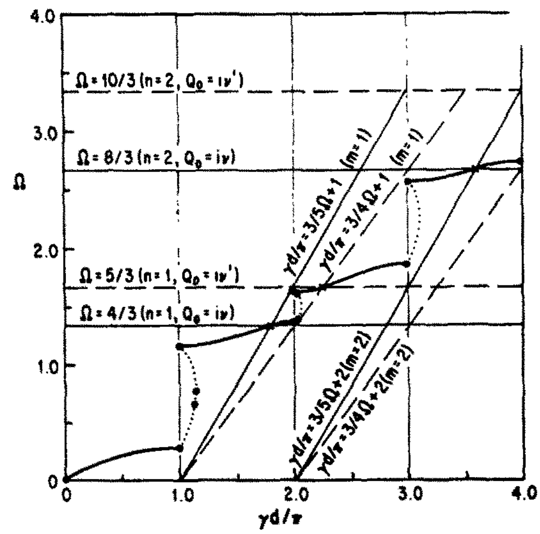

Fig. 1. Dispersion curve with bounds shown for $Q_0 = i\nu$(—) and $i\omega'$(— — —). End points are shown as ●, intersection of bounds as x and complex roots as ⊙.

The scale for Ω in Ref. [1] is different from that used in this paper by a factor of 2 since ω there was nondimensionalized with respect to the lowest thickness shear mode frequency of an infinite plate of half width a while here a half width h, equal to $a/2$ for a solid cylinder, is used. Thus values from Ref. [1] used for comparison here must be divided by 2. Thus, the cut off frequencies are at $\Omega = 0$, 0.290, 1.168, 1.359, 1.628, 1.872, 2.620, 2.772, etc. The intersection points of the bounding curves lie for $Q_0 = i\nu$ at (1.8, 1.33) for $\eta = 1$, $m = 1$ and at (3.6, 2.67) for $\eta = 2$, $m = 2$ while for $Q_0 = i\nu'$, the first point lies at (2.25, 1.67) for $\eta = 1$, $m = 1$ and the others lie beyond the range of calculated values. The complex roots lie at $(1 + 0.177i, 0.764)$ and $(1 + 0.729i\ 0.679)$ for $Q_0 = 0, \infty$ respectively on the first complex branch, at $(2 + 0.055i, 1.361)$ and $(2 + 0.080i, 1.1624)$ for $Q_0 = 0, \infty$ respectively on the second branch, etc.

The effort in calculating all of these bounds and solution points and the end points of the Brillouin zones is considerably less than that involved in determining the complete dispersion spectrum, even for this simple problem, yet these few calculations are sufficient for a reasonably accurate sketch. It is anticipated that this reduction in effort will apply to more general problems as well.

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